TEST IN CALCULUS

SOLUTIONS

QUESTION 1

$$\lim_{x\to 0} \frac{e^x - 1 - x - 0.5x^2}{x^3} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{e^x - 1 - x}{3x^2}$$

$$= \lim_{x\to 0} \frac{e^x - 1}{6x} = \lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6}$$

QUESTION 2

(a) $$\frac{dy}{dx} + y\tan x = e^{3x}\cos x$$

$$P(x) = \tan x \qquad Q(x) = e^{3x}\cos x$$

Integrating factor:

$$I = e^{\int P(x)\,dx} = e^{-\ln(\cos x)} = \frac{1}{\cos x}$$

Thus

$$Iy = \int IQ\,dx \Rightarrow \frac{y}{\cos x} = \int e^{3x}\,dx$$

$$\Rightarrow \frac{y}{\cos x} = \frac{e^{3x}}{3} + c \Rightarrow y = \left(\frac{e^{3x}}{3} + c\right)\cos x$$

$$y(0) = 1 \Rightarrow \left(\frac{1}{3} + c\right)\cos 0 = 1 \Rightarrow c = \frac{2}{3}$$

The particular solution is

$$y = \left(\frac{e^{3x}}{3} + \frac{2}{3}\right)\cos x$$

(b)

n	x_n	y_n
0	0	1
1	0.2	1.2
2	0.4	1.50851
3	0.6	1.99256
4	0.8	2.71852
5	1	3.69469

The approximation is $y \cong 3.70$

(c) The exact solution for $x=1$ (found in (a)) is

$$y = \left(\frac{e^3}{3} + \frac{2}{3}\right)\cos 1 = 3.97762$$

$$\text{error} = |3.97762 - 3.70| = 0.277 < 0.3$$

QUESTION 3

(a) $\dfrac{dy}{dx} = \dfrac{36x^2 + 13xy + y^2}{x^2} \Rightarrow \dfrac{dy}{dx} = 36 + 13\dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2$

(b) Let $u = \dfrac{y}{x} \Rightarrow y = ux \Rightarrow \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

Hence

$$u + x\frac{du}{dx} = 36 + 13u + u^2 \Rightarrow x\frac{du}{dx} = 36 + 12u + u^2$$

$$\Rightarrow x\frac{du}{dx} = (u+6)^2 \Rightarrow \int \frac{du}{(u+6)^2} = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{u+6} = \ln x + c \Rightarrow u + 6 = -\frac{1}{\ln x + c}$$

$$\Rightarrow \frac{y}{x} = -\frac{1}{\ln x + c} - 6 \Rightarrow y = -\frac{x}{\ln x + c} - 6x$$

$$y(1) = 1 \Rightarrow -\frac{1}{c} - 6 = 1 \Rightarrow -\frac{1}{c} = 7 \Rightarrow c = -\frac{1}{7}$$

Therefore

$$y = \frac{x}{\frac{1}{7} - \ln x} - 6x$$

QUESTION 4

(a) $\sum_{n=1}^{\infty} \frac{2n+1}{3n+4}$ diverges by divergence test

since $\lim_{n\to\infty} \frac{2n+1}{3n+4} = \frac{2}{3} \neq 0$

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ We apply RATIO TEST

$$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{(2n+2)!}{((n+1)!)^2} \frac{(n!)^2}{(2n)!}$$

$$= \lim_{n\to+\infty} \frac{(2n+1)(2n+2)}{(n+1)^2} = 4 > 1$$

Therefore, the series diverges

QUESTION 5

(a) $\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx$

$$= -\frac{x}{e^x} - \frac{1}{e^x} + C$$

Thus

$$\int_1^{\infty} x e^{-x} dx = \lim_{a \to +\infty} \left[-\frac{x}{e^x} - \frac{1}{e^x} \right]_1^a$$

$$= \lim_{a \to +\infty} \left(\frac{1}{e} + \frac{1}{e} - \frac{a}{e^a} - \frac{1}{e^a} \right)$$

$$= \lim_{a \to \infty} \left(\frac{2}{e} - \frac{a}{e^a} - \frac{1}{a} \right) = \frac{2}{e}$$

since $\lim_{a \to \infty} \frac{1}{e^a} = 0$ and $\lim_{a \to \infty} \frac{a}{e^a} \overset{\frac{\infty}{\infty}}{=} \lim_{a \to \infty} \frac{1}{e^a} = 0$

(b) (i) $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges by integral test and (a).

(ii) If $a_n = \frac{n^2}{(2n+1)e^n}$ and $b_n = \frac{n}{e^n}$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2}{(2n+1)e^n} \cdot \frac{e^n}{n} = \frac{1}{2}$$

Since $\sum b_n$ converges $\sum a_n$ converges as well.

QUESTION 6

(a) $\sum_{n=1}^{\infty} \frac{(0.4)^n}{n} x^n$ RATIO TEST

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{0.4^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{0.4^n \cdot x} \right|$$

$$= \lim_{n\to\infty} \frac{0.4n}{n+1} |x| = 0.4|x|$$

$0.4R = 1 \Rightarrow R = 2.5$ RADIUS OF CONVERGGENCE

(b) Thus, the series certainly converges within $x \in]-2.5, 2.5[$

For $x = 2.5$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

For $x = -2.5$ $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges

since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n}$ decreases.

Interval of convergence $x \in [-2.5, 2.5[$

(c) The center of convergence is $c = 3$.

Interval of convergence $x \in [0.5, 5.5[$.