HAEF IB – FURTHER MATH HL

TEST 2

SETS, RELATIONS AND GROUPS

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SOLUTIONS

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Questions

1. *[Maximum mark: 10]*

The binary operation $*$ is defined on $\mathbb{N}$ by $a*b=1+ab$.

Determine whether or not $*$

(a) is closed; *[2 marks]*

(b) is commutative; *[2 marks]*

(c) is associative; *[3 marks]*

(d) has an identity element. *[3 marks]*

Solution

(a) $*$ is closed **A1**
because $1+ab\in\mathbb{N}$ (when $a,b\in\mathbb{N}$) **R1**

[2 marks]

(b) consider
$a*b=1+ab=1+ba=b*a$ **M1A1**
therefore $*$ is commutative

[2 marks]

(c) **EITHER**

$a*(b*c)=a*(1+bc)=1+a(1+bc)\ \ (=1+a+abc)$ **A1**
$(a*b)*c=(1+ab)*c=1+c(1+ab)\ \ (=1+c+abc)$ **A1**
(these two expressions are unequal when $a\neq c$) so $*$ is not associative **R1**

OR

proof by counter example, for example
$1*(2*3)=1*7=8$ **A1**
$(1*2)*3=3*3=10$ **A1**
(these two numbers are unequal) so $*$ is not associative **R1**

[3 marks]

(d) let e denote the identity element; so that
$a*e=1+ae=a$ gives $e=\dfrac{a-1}{a}$ (where $a\neq 0$) **M1**
then any valid statement such as: $\dfrac{a-1}{a}\notin\mathbb{N}$ or e is not unique **R1**
there is therefore no identity element **A1**

Note: Award the final ***A1*** only if the previous ***R1*** is awarded.

[3 marks]

2. *[Maximum mark: 8]*

The elements of sets P and Q are taken from the universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. $P = \{1, 2, 3\}$ and $Q = \{2, 4, 6, 8, 10\}$.

(a) Given that $R = (P \cap Q')'$, list the elements of R. *[3 marks]*

(b) For a set S, let S^* denote the set of all subsets of S,

(i) find P^*;

(ii) find $n(R^*)$. *[5 marks]*

Solution

(a) $P = \{1, 2, 3\}$

$Q' = \{1, 3, 5, 7, 9\}$

so $P \cap Q' = \{1, 3\}$ ***(M1)(A1)***

so $(P \cap Q')' = \{2, 4, 5, 6, 7, 8, 9, 10\}$ ***A1***

[3 marks]

(b) (i) $P^* = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}, \varnothing\}$ ***A2***

Note: Award ***A1*** if one error, ***A0*** for two or more.

(ii) R^* contains: the empty set (1 element); sets containing one element (8 elements); sets containing two elements ***(M1)***

$= \binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \ldots \binom{8}{8}$ ***(A1)***

$= 2^8 \; (= 256)$ ***A1***

Note: ***FT*** in (ii) applies if no empty set included in (i) and (ii).

[5 marks]

3. *[Maximum mark: 13]*

The function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases}2x+1 & \text{for } x\le 2\\ x^2-2x+5 & \text{for } x>2\,.\end{cases}$$

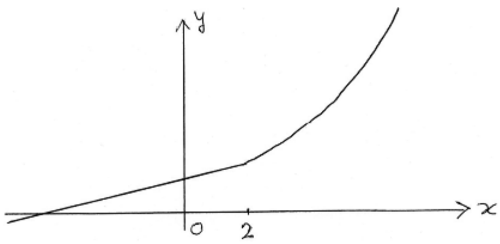
(a) (i) Sketch the graph of f.

(ii) By referring to your graph, show that f is a bijection. *[5 marks]*

(b) Find $f^{-1}(x)$. *[8 marks]*

Solution

(a) (i)



A1A1

(ii) demonstrating the need to show that f is both an injection and a surjection (seen anywhere) ***(R1)***

f is an injection by any valid reason *eg* horizontal line test, strictly increasing function ***R1***

the range of f is $\mathbb{R}$ so that f is a surjection ***R1***

f is therefore a bijection ***AG***

[5 marks]

(b) considering the linear section, put

$y=2x+1$ or $x=2y+1$ ***(M1)***

$x=\dfrac{y-1}{2}$ or $y=\dfrac{x-1}{2}$ ***A1***

so $f^{-1}(x)=\dfrac{x-1}{2},\ x\le 5$ ***A1***

EITHER

$y=(x-1)^2+4$ ***M1A1***

$(x-1)^2=y-4$

$x=1\pm\sqrt{y-4}$ ***A1***

$x=1+\sqrt{y-4}$

taking the + sign to give the right hand half of the parabola ***R1***

so $f^{-1}(x)=1+\sqrt{x-4},\ x>5$ ***A1***

OR

considering the quadratic section, put

$y=x^2-2x+5$

$x^2-2x+5-y=0$ ***M1***

$x=\dfrac{2\pm\sqrt{4-4(5-y)}}{2}\ (=1\pm\sqrt{y-4})$ ***M1A1***

taking the + sign to give the right hand half of the parabola ***R1***

so $f^{-1}(x)=\dfrac{2+\sqrt{4-4(5-x)}}{2},\ x>5\ (f^{-1}(x)=1+\sqrt{x-4},\ x>5)$ ***A1***

[8 marks]

4. *[Maximum mark: 13]*

The relation R is defined on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ by aRb if and only if $a(a+1) \equiv b(b+1)(\text{mod}\,5)$.

(a) Show that R is an equivalence relation. *[6 marks]*

(b) Show that the equivalence defining R can be written in the form

$$(a-b)(a+b+1) \equiv 0(\text{mod}\,5).$$ *[3 marks]*

(c) Hence, or otherwise, determine the equivalence classes. *[4 marks]*

Solution

(a) reflexive: $a(a+1) \equiv a(a+1)(\text{mod}\,5)$, therefore aRa ***R1***

symmetric: $aRb \Rightarrow a(a+1) = b(b+1) + 5N$ ***M1***

$\Rightarrow b(b+1) = a(a+1) - 5N \Rightarrow bRa$ ***A1***

transitive:

EITHER

aRb and $bRc \Rightarrow a(a+1) = b(b+1) + 5M$ and $b(b+1) = c(c+1) + 5N$ ***M1***

it follows that $a(a+1) = c(c+1) + 5(M+N) \Rightarrow aRc$ ***M1A1***

OR

aRb and $bRc \Rightarrow a(a+1) \equiv b(b+1)(\text{mod}\,5)$ and

$b(b+1) \equiv c(c+1)(\text{mod}\,5)$. ***M1***

$a(a+1) - b(b+1) \equiv 0(\text{mod}\,5)$; $b(b+1) - c(c+1) \equiv 0(\text{mod}\,5)$ ***M1***

$a(a+1) - c(c+1) \equiv 0 \,\text{mod}\, 5 \Rightarrow a(a+1) \equiv c(c+1) \,\text{mod}\, 5 \Rightarrow aRc$ ***A1***

[6 marks]

(b) the equivalence can be written as

$a^2 + a - b^2 - b \equiv 0(\text{mod}\,5)$ ***M1***

$(a-b)(a+b) + a - b \equiv 0(\text{mod}\,5)$ ***M1A1***

$(a-b)(a+b+1) \equiv 0(\text{mod}\,5)$ ***AG***

[3 marks]

(c) the equivalence classes are

$\{1, 3, 6, 8, 11\}$

$\{2, 7, 12\}$

$\{4, 5, 9, 10\}$ ***A4***

Note: Award ***A3*** for 2 correct classes, ***A2*** for 1 correct class.

[4 marks]

5. *[Maximum mark: 10]*

(a) The function $g:\mathbb{Z}\to\mathbb{Z}$ is defined by $g(n)=|n|-1$ for $n\in\mathbb{Z}$. Show that g is neither surjective nor injective. *[2 marks]*

(b) The set S is finite. If the function $f:S\to S$ is injective, show that f is surjective. *[2 marks]*

(c) Using the set $\mathbb{Z}^+$ as both domain and codomain, give an example of an injective function that is not surjective. *[3 marks]*

(d) Using the set Z^+ as both domain and codomain, give an example of a surjective function that is not injective. *[3 marks]*

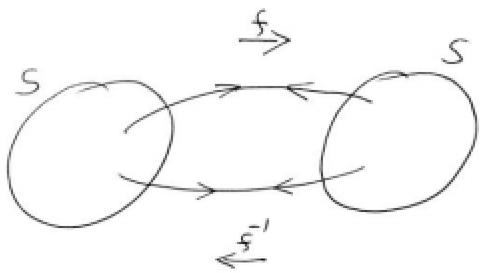
Solution

(a) non-S: for example -2 does not belong to the range of g ***R1***
non-I: for example $g(1)=g(-1)=0$ ***R1***

Note: Graphical arguments have to recognize that we are dealing with sets of integers and not all real numbers

[2 marks]

(b) as f is injective $n\big(f(S)\big)=n(S)$ ***A1***



R1

Note: Accept alternative explanations.

f is surjective ***AG***

[2 marks]

(c) for example, $h(n)=n+1$ ***A1***

Note: Only award the ***A1*** if the function works.

I: $n+1=m+1\Rightarrow n=m$ ***R1***
non-S: 1 has no pre-image as $0\notin\mathbb{Z}^+$ ***R1***

[3 marks]

(d) $f(x)=\begin{cases}1, & n=1\\ n-1, & n>1\end{cases}$

Surjective as range = codomain

Non-injective as $f(1)=1=f(2)$

6. *[Maximum mark: 12]*

The binary operation Δ is defined on the set $S = \{1, 2, 3, 4, 5\}$ by the following Cayley table.

Δ	**1**	**2**	**3**	**4**	**5**
1	1	1	2	3	4
2	1	2	1	2	3
3	2	1	3	1	2
4	3	2	1	4	1
5	4	3	2	1	5

(a) State whether S is closed under the operation Δ and justify your answer. *[2]*

(b) State whether Δ is commutative and justify your answer. *[2]*

(c) State whether there is an identity element and justify your answer. *[2]*

(d) Determine whether Δ is associative and justify your answer. *[3]*

(e) Find the solutions of the equation $a\Delta b = 4\Delta b$, for $a \neq 4$. *[3]*

Solution

(a) yes **A1**
because the Cayley table only contains elements of S **R1**

[2 marks]

(b) yes **A1**
because the Cayley table is symmetric **R1**

[2 marks]

(c) no **A1**
because there is no row (and column) with 1, 2, 3, 4, 5 **R1**

[2 marks]

(d) attempt to calculate $(a\Delta b)\Delta c$ and $a\Delta(b\Delta c)$ for some $a,b,c\in S$ **M1**
counterexample: for example, $(1\Delta 2)\Delta 3 = 2$
$1\Delta(2\Delta 3) = 1$ **A1**
Δ is not associative **A1**

> **Note:** Accept a correct evaluation of $(a\Delta b)\Delta c$ and $a\Delta(b\Delta c)$ for some $a,b,c\in S$ for the ***M1***.

[3 marks]

(e) for example, attempt to enumerate $4\Delta b$ for $b = 1, 2, 3, 4, 5$ and obtain
$(3, 2, 1, 4, 1)$ ***(M1)***
find $(a, b) \in \{(2, 2), (2, 3)\}$ for $a \neq 4$ (or equivalent) ***A1A1***

> **Note:** Award ***M1A1A0*** if extra 'solutions' are listed.

[3 marks]

7. *[Maximum mark: 19]*

Consider the set S defined by $S = \{s \in \mathbb{Q} : 2s \in \mathbb{Z}\}$.

You may assume that $+$ (addition) and $\times$ (multiplication) are associative binary operations on $\mathbb{Q}$.

(a) (i) Write down the six smallest non-negative elements of S.

(ii) Show that $\{S, +\}$ is a group.

(iii) Give a reason why $\{S, \times\}$ is not a group. Justify your answer. *[9]*

(b) The relation R is defined on S by $s_1 R s_2$ if $3s_1 + 5s_2 \in \mathbb{Z}$.

(i) Show that R is an equivalence relation.

(ii) Determine the equivalence classes. *[10]*

Solution

(a) (i) $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ ***A2***

(ii) closure: if $s_1, s_2 \in S$, then $s_1 = \frac{m}{2}$ and $s_2 = \frac{n}{2}$ for some $m, n \in$ ¢ . ***M1***

$s_1 + s_2 = \frac{m+n}{2} \in S$ ***A1***

OR the sum of two half-integers is a half integer ***A1R1***

THEN

identity: 0 is the (additive) identity ***A1***

inverse: $s + (-s) = 0$, where $-s \in S$ ***A1***

it is associative (since $S \subset$ §) ***A1***

the group axioms are satisfied ***AG***

(iii) **EITHER**

the set is not closed under multiplication, ***A1***

for example, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, but $\frac{1}{4} \notin S$ ***R1***

OR

not every element has an inverse, ***A1***

for example, 3 does not have an inverse ***R1***

[9 marks]

(b) (i) reflexive: consider $3s + 5s$ ***M1***

$= 8s \in$ ¢ $\Rightarrow$ reflexive ***A1***

symmetric: if $s_1 R s_2$, consider $3s_2 + 5s_1$ ***M1***

for example, $= 3s_1 + 5s_2 + (2s_1 - 2s_2) \in$ ¢ $\Rightarrow$ symmetric ***A1***

transitive: if $s_1 R s_2$ and $s_2 R s_3$, consider ***(M1)***

$3s_1 + 5s_3 = (3s_1 + 5s_2) + (3s_2 + 5s_3) - 8s_2$ ***M1***

$\in$ ¢ $\Rightarrow$ transitive ***A1***

so R is an equivalence relation ***AG***

(ii) $C_1 =$ ¢ ***A1***

$C_2 = \left\{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots\right\}$ ***A1A1***

Note: ***A1*** for half odd integers and ***A1*** for $\pm$.

[10 marks]

8. *[Maximum mark: 15]*

Sets X and Y are defined by $X =]0, 1[$; $Y = \{0, 1, 2, 3, 4, 5\}$.

(a) (i) Sketch the set $X \times Y$ in the Cartesian plane.

(ii) Sketch the set $Y \times X$ in the Cartesian plane.

(iii) State $(X \times Y) \cap (Y \times X)$. *[5]*

Consider the function $f: X \times Y \to \mathbb{R}$ defined by $f(x, y) = x + y$

and the function $g: X \times Y \to \mathbb{R}$ defined by $g(x, y) = xy$.

(b) (i) Find the range of the function f.

(ii) Find the range of the function g.

(iii) Show that f is an injection.

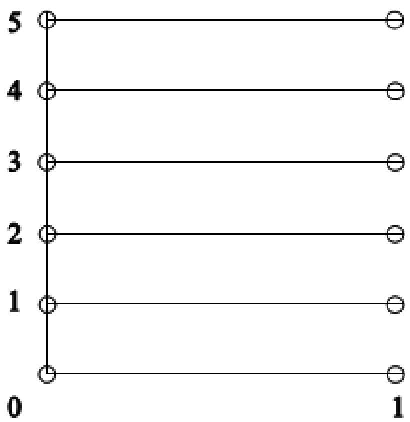
(iv) Find $f^{-1}(\pi)$, expressing your answer in exact form.

(iv) Find $f^{-1}(\pi)$, expressing your answer in exact form.

(v) Find all solutions to $g(x, y) = \frac{1}{2}$. *[10]*

Solution

(a) (i)

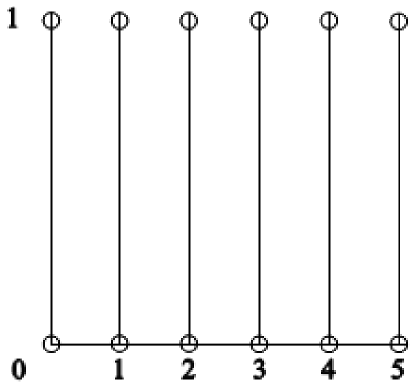


correct horizontal lines ***A1***

correctly labelled axes ***A1***

clear indication that the endpoints are not included ***A1***

(ii)



fully correct diagram ***A1***

Note: Do not penalize the inclusion of endpoints twice.

(iii) the intersection is empty ***A1***

[5 marks]

(b) (i) range $(f) =]0, 1[\cup]1, 2[\cup \ldots \cup]5, 6[$ ***A1A1***

Note: ***A1*** for six intervals and ***A1*** for fully correct notation. Accept $0 < x < 6$, $x \neq 0, 1, 2, 3, 4, 5, 6$.

(ii) range $(g) = [0, 5[$ ***A1***

(iii) Attempt at solving
$f(x_1, y_1) = f(x_2, y_2)$ ***M1***

$f(x, y) \in]y, y+1[\Rightarrow y_1 = y_2$ ***M1***
and then $x_1 = x_2$ ***A1***
so f is injective ***AG***

(iv) $f^{-1}(\pi) = (\pi - 3, 3)$ ***A1A1***

(v) solutions: $(0.5, 1), (0.25, 2), \left(\frac{1}{6}, 3\right), (0.125, 4), (0.1, 5)$ ***A2***

Note: ***A2*** for all correct, ***A1*** for 2 correct.

[10 marks]