

# HAEF IB – FURTHER MATH HL

## TEST 3

### NUMBER THEORY

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Name: **SOLUTIONS**

Date: **16-2-2018**

Marks: \_\_\_\_\_ /90

Grade: \_\_\_\_\_

### Questions

1. [Maximum mark: 7]

- (a) Explain why  $23!+5$  is not a prime number. [1 mark]  
(b) Find all positive integers  $n$  for which  $n!+5$  is a prime number. [3 marks]  
(c) Find 2018 consecutive integers which are not prime. [3 marks]

2. [Maximum mark: 8]

Let  $a$  and  $b$  be positive integers. Show that

- (a) If  $a+b$  and  $2a-b$  are coprime then  $a$  and  $b$  are coprime. [2 marks]  
(b) If  $a$  and  $b$  are coprime then  $\gcd(a+b,2a-b)$  is either 1 or 3. [4 marks]  
(c) Show by giving examples that both the results in (b) are possible. [2 marks]

3. [Maximum mark: 8]

- (a) Find  $2018^{2018} \pmod{13}$  [4 marks]  
(b) Find the last digit of  $2018^{2018}$  [4 marks]

4. [maximum mark: 5]

Show that there are infinitely many primes.

**5.** [maximum mark: 7]

Solve **analytically** the system of congruences

$$x \equiv 1 \pmod{3}$$

$$x \equiv 2 \pmod{4}$$

$$x \equiv 3 \pmod{5}$$

$$x \equiv 4 \pmod{7}$$

**6.** [maximum mark: 6]

Solve the difference equation

$$u_{n+1} = 8u_n - 16u_{n-1}$$

$$u_0 = 3, \quad u_1 = 16$$

**7.** [maximum mark: 5]

Show that an integer  $a$  is divisible by 3 if the sum of the digits in the expression of  $a$  in base 7 is divisible by 3.

**8.** [maximum mark: 12]

Let  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$

(a) Show that

$$a + c \equiv b + d \pmod{m} \quad [2 \text{ marks}]$$

$$ac \equiv bd \pmod{m} \quad [4 \text{ marks}]$$

(a) Show by using mathematical induction that

$$a^n \equiv b^n \pmod{m} \quad \text{for any } n \in \mathbb{Z}^+ \quad [6 \text{ marks}]$$

**9.** [maximum mark: 10]

Consider the non-homogeneous difference equation

$$u_{n+2} = 5u_{n+1} - 6u_n + 10$$

$$u_1 = 8, \quad u_2 = 30$$

(a) By letting  $V_n = u_n - 5$ , find the first 3 terms of the sequence  $V_n$  [2 marks]

(b) Show that

$$V_{n+2} = 5V_{n+1} - 6V_n \quad [2 \text{ marks}]$$

(c) Find the general term for  $V_n$  and hence for  $u_n$ . [6 marks]

**10.** [maximum mark: 22]

Consider

$$d_1 = \gcd(88, 136)$$

$$d_2 = \gcd(88, 137)$$

$$d_3 = \gcd(88, 138)$$

- (a) Find the prime decomposition of the numbers 88 and 136 and hence

write down the values of  $d_1$  and  $l_1 = \text{lcm}(88, 136)$  [4 marks]

- (b) Given that  $9 \times 137 - 14 \times 88 = 1$ , explain why  $d_2 = 1$  [2 marks]

- (c) Find the value of  $d_3$  by using Euclid's algorithm; Hence express  $d_3$  as a linear combination of 88 and 138. [5 marks]

- (d) Solve each of the following congruences

(i)  $88x \equiv 2 \pmod{136}$ . [1 mark]

(ii)  $88x \equiv 2 \pmod{137}$ . [2 marks]

(iii)  $88x \equiv 2 \pmod{138}$ . [4 marks]

- (e) Find the general solution of the Diophantine equation

$$88x - 138y = 2 \quad [4 \text{ marks}]$$

# NUMBER THEORY

16-2-2018

## SOLUTIONS

1. (a)  $5/23!$  and  $5/5 \Rightarrow 5/23!+5$

(b) For  $n \geq 5$   $5/n!+5$  composite

For  $n=1$   $1!+5=6$  composite

For  $n=2$   $2!+5=7$  prime

For  $n=3$   $3!+5=11$  prime

For  $n=4$   $4!+5=29$  prime

Thus for  $n=2, 3, 4$

(c)  $2019!+2, 2019!+3, \dots, 2019!+2019$

are all composite

2. (a) Proof 1

Let  $d_1 = \gcd(a, b)$  and  $d_2 = \gcd(a+b, 2a-b)$

$d_1/a$  and  $d_1/b \Rightarrow d_1/a+b$  and  $d_1/2a-b$   
 $\Rightarrow d_1/d_2$

But  $d_2=1$ , thus  $d_1=1$  i.e.  $a, b$  coprime

Proof 2

$a+b, 2a-b$  coprime  $\Rightarrow r(a+b)+s(2a-b)=1$ , for some  $r, s$   
 $\Rightarrow (r+2s)a+(r-s)b=1$   
 $\Rightarrow a, b$  coprime

(b) Let  $d_1, d_2$  as above

$d_2/a+b$  and  $d_2/2a-b \Rightarrow d_2/a+b+2a-b \Rightarrow d_2/3a$

Similarly  $d_2/2(a+b)-(2a-b) \Rightarrow d_2/3b$

Since  $a, b$  coprime  $d_2/3$  or  $d_2/a, b \Rightarrow d_2=3$  or  $d_2=1$

$$(c) \quad a=5, b=3 \quad \text{give} \quad d_2 = \gcd(8, 7) = 1$$

$$a=7, b=5 \quad \text{give} \quad d_2 = \gcd(12, 9) = 3$$

3. (a)  $2018^{12} \equiv 1 \pmod{13}$  by Fermat.

$$2018^{12 \cdot 168} \equiv 1 \pmod{13}$$

$$2018^{2016} \equiv 1 \pmod{13}$$

$$2018^{2018} \equiv 2018^2 \pmod{13}$$

$$\equiv 3^2 \pmod{13} \quad (\text{since } 2018 \equiv 3 \pmod{13})$$

$$\equiv 9 \pmod{13}$$

(b) Look for  $2018^{2018} \pmod{10}$

$$2018 \equiv 3 \pmod{5}$$

$$2018^{2018} \equiv 3^{2018} \pmod{5}$$

$$\equiv 3^{4 \times 504 + 2} \pmod{5}$$

$$\equiv 3^2 \pmod{5} \quad \text{since } 3^4 \equiv 1 \pmod{5}$$

$$\equiv 4 \pmod{5}$$

$$\text{Since } 2018^{2018} \equiv 4 \pmod{5} \Rightarrow 2018^{2018} \equiv 4 \pmod{10}$$

Thus the last digit is 4.

4. Suppose that there are finitely many primes  $p_1, p_2, \dots, p_n$ . Then

$$S = p_1 p_2 \cdots p_n + 1$$

has a prime divisor, say  $p_i$ . Then  $p_i | S$

$$p_i | S, p_i | p_1 \cdots p_n \Rightarrow p_i | 1 \text{ contradiction}$$

$$5. \quad x \equiv 1 \pmod{3} \Rightarrow x = 3a + 1$$

$$x = 3a + 1 \equiv 2 \pmod{4}$$

$$\Rightarrow 3a \equiv 1 \pmod{4}$$

$$\Rightarrow a \equiv 3 \pmod{4} \Rightarrow a = 4b + 3$$

$$\text{so } x = 3(4b+3) + 1 = 12b + 10$$

$$x = 12b + 10 \equiv 3 \pmod{5}$$

$$\Rightarrow 12b \equiv 3 \pmod{5}$$

$$\Rightarrow 4b \equiv 1 \pmod{5}$$

$$\Rightarrow b \equiv 4 \pmod{5} \Rightarrow b = 5c + 4$$

$$\text{so } x = 12(5c+4) + 10 = 60c + 58$$

$$x = 60c + 58 \equiv 4 \pmod{7}$$

$$\Rightarrow 60c \equiv -54 \pmod{7}$$

$$\Rightarrow 4c \equiv 2 \pmod{7} \quad (\text{since } 60 \equiv 4 \pmod{7})$$

$$\Rightarrow c \equiv 4 \pmod{7}$$

$$\text{For } c=1, \quad x = 60c + 58$$

The unique solution is  $x = 298 \pmod{3 \cdot 4 \cdot 5 \cdot 7}$   
 $\underline{x = 298 \pmod{420}}$

6. The auxiliary equation is

$$r^2 - 8r + 16 \Rightarrow r^2 - 8r + 16 = 0 \Rightarrow (r-4)^2 = 0 \Rightarrow r = 4 \text{ (double)}$$

The solution has the form

$$u_n = a \cdot 4^n + b \cdot n \cdot 4^n$$

$$\text{For } n=0 : \boxed{a = 3}$$

$$\text{For } n=1 : 4a + 4b = 16 \Rightarrow 4b = 4 \Rightarrow \boxed{b = 1}$$

$$\text{Therefore, } u_n = 3 \cdot 4^n + n \cdot 4^n$$

$$7. \text{ Let } a = a_n \gamma^n + a_{n-1} \gamma^{n-1} + \dots + a_1 \gamma + a_0.$$

$$\text{Since } \gamma \equiv 1 \pmod{3}$$

$$\Rightarrow \gamma^k \equiv 1^k \pmod{3}$$

$$\Rightarrow \gamma^k \equiv 1 \pmod{3}$$

$$\text{Hence, } a \equiv a_n + a_{n-1} + \dots + a_1 + a_0 \pmod{3}$$

$$\text{Therefore } 3 | a \Leftrightarrow 3 | a_n + a_{n-1} + \dots + a_0$$

$\Leftrightarrow 3 | \text{sum of digits}$

$$8. (a) (i) \left. \begin{array}{l} m | a-b \\ m | c-d \end{array} \right\} \Rightarrow m | a-b+c-d = (a+d)-(b+d)$$

$$\Rightarrow a+c \equiv b+d \pmod{m}$$

$$(ii) \left. \begin{array}{l} m | a-b \\ m | c-d \end{array} \right\} \Rightarrow m | (a-b) \cdot (c-d)$$

$$\Rightarrow m | ac-ad-bc+bd$$

$$\Rightarrow m | ac-bd+bd-ad-bc+bd$$

$$\Rightarrow m | ac-bd + d(b-a) + b(c-d)$$

$$\Rightarrow m | ac-bd$$

$$\Rightarrow ac \equiv bd \pmod{m}$$

(b) For  $n=1$ , it is trivial.

Assume it is true for  $n=k$ , i.e.  $a^k \equiv b^k \pmod{m}$

Prove it is true for  $n=k+1$ , i.e.  $a^{k+1} \equiv b^{k+1} \pmod{m}$

$$\text{Indeed, by (a)(ii)} \quad a^k \cdot a \equiv b^k \cdot b \pmod{m}$$

$$\Rightarrow a^{k+1} \equiv b^{k+1} \pmod{m}$$

Therefore, by mathematical induction, the statement is true for any  $n \in \mathbb{Z}^+$

$$9. (a) u_1 = 8, u_2 = 30, u_3 = 112$$

$$\text{Hence } V_1 = 3, V_2 = 25, V_3 = 107$$

$$(b) u_n = V_n + 5, \text{ hence}$$

$$u_{n+2} = 5u_{n+1} - 6u_n + 10$$

$$\Rightarrow V_{n+2} + 5 = 5(V_{n+1} + 5) - 6(V_n + 5) + 10$$

$$\Rightarrow V_{n+2} + 5 = 5V_{n+1} + 25 - 6V_n - 30 + 10$$

$$\Rightarrow V_{n+2} = 5V_{n+1} - 6V_n$$

(c) The auxiliary equation is

$$r^2 - 5r + 6 = 0 \Rightarrow r = 2 \text{ or } r = 3$$

The general solution has the form

$$V_n = a \cdot 2^n + b \cdot 3^n$$

$$\begin{aligned} \text{For } n=1 \quad 2a + 3b &= 3 \\ \text{For } n=2 \quad 4a + 9b &= 25 \end{aligned} \quad \left. \begin{array}{l} \end{array} \right\} \Rightarrow a = -8, b = \frac{19}{3}$$

$$\text{Hence } V_n = -8 \cdot 2^n + \frac{19}{3} \cdot 3^n = -2^{n+3} + 19 \cdot 3^{n-1}$$

Therefore

$$u_n = V_n + 5 = 19 \cdot 3^{n-1} - 2^{n+3} + 5$$

$$10. (a) \quad 88 = 2^3 \cdot 11 \quad 136 = 2^3 \cdot 17$$

$$d_1 = \gcd(88, 136) = 2^3 = 8$$

$$l_1 = \text{lcm}(88, 136) = 2^3 \cdot 11 \cdot 17 = 1496$$

$$(b) \quad d_2 \mid 88 \text{ and } d_2 \mid 136 \Rightarrow d_2 \mid 1 \Rightarrow d_2 = 1$$

$$(c) \quad 138 = 1 \cdot 88 + 50$$

$$88 = 1 \cdot 50 + 38$$

$$50 = 1 \cdot 38 + 12$$

$$38 = 3 \cdot 12 + 2$$

$$12 = 6 \cdot 2 + 0$$

$$\gcd(138, 88) = 2$$

$$2 = 38 - 3 \cdot 12 = 38 - 3(50 - 38)$$

$$= 1 \cdot 38 - 3 \cdot 50$$

$$= 1 \cdot (88 - 50) - 3 \cdot 50$$

$$= 1 \cdot 88 - 7 \cdot 50$$

$$= 1 \cdot 88 - 7(138 - 88)$$

$$= 11 \cdot 88 - 7 \cdot 138$$

(d) (i) No solution since  $d_1 = 8 \nmid 2$

(ii) Unique solution since  $d_2 = 1 \mid 2$

$$9 \times 137 - 14 \times 88 = 1 \Rightarrow 18 \times 137 - 28 \times 88 = 2$$

$$\text{Solution } x \equiv -28 \pmod{137} \equiv \underline{109 \pmod{137}}$$

(iii) Two solutions since  $d_3 = 2 \mid 2$

$$88x \equiv 2 \pmod{138} \Rightarrow 44x \equiv 1 \pmod{69}$$

$$\text{But } 2 = 11 \cdot 88 - 7 \cdot 138 \Rightarrow 1 = 11 \cdot 44 - 7 \cdot 69$$

$$\text{Solutions: } x \equiv \underline{11 \pmod{138}}, \quad x \equiv 11 + 69 \equiv \underline{80 \pmod{138}}$$

$$(e) \quad 88x - 138y = 2. \quad \text{But } 88 \times 11 - 138 \times 7 = 2$$

$$\text{Particular solution } (x_0, y_0) = (11, 7)$$

$$\text{General solution } (x_0 + \frac{138}{2}t, y_0 + \frac{88}{2}t) = (11 + 69t, 7 + 44t)$$